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Introduction

The well-known algebraic structure is residuated lattice. BL-algebras, MTL-algebras, MV-algebras, and so forth, are the best known classes of residuated lattices [4, 5]. Since the algebra of truth values is no longer a residuated lattice, a specific algebra is introduced and called an EQ-algebra [11] by Novák and De Baets. EQ-algebras generalize the residuated lattices which have three binary operations, meet, multiplication, fuzzy equality, and a unit element. As it was mentioned in [7], if the product operation in EQ-algebras is replaced by a other binary operation smaller or equal than the original product we will still obtain an EQ-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, a new structure, called equality algebra similar to EQ-algebras, but without a product, was introduced by S. Jenei in [1, 7]. An equality algebra has tow basic binary operations (meet, equality) and a top element 1.

Filter theory plays an important role in studying these algebras. From a logical point of view, various filters correspond to various sets of provable formulas. The sets of provable formulas can be described by fuzzy filters of those algebraic semantics. Up to now, some types of filters on special residuated lattices based on logical algebras have been widely studied and some important results have been obtained, [1, 2, 6, 12, 13]. In [10, 11], the notions of a prefilter (which coincides with filters in residuated lattices), a prime prefilter, and a (positive) implicative prefilter in EQ-algebras were proposed and some characterizations of them have been investigated.

In the first Chapter, we give a basic notions and definitions on ordered sets and equality algebra and there proprieties needed in the sequel and we present some type of filters.

In the second Chapter, we introduce some important results concerning positive implicative filters and implicative filters in equality algebra.

In the third Chapter, we present some results concerning fantastic and Boolean filters in equality algebra and we give some examples.

Chapter 1

Preliminaries

Abstract

In this Chapter, we recall some basic notions and definitions on ordered sets needed in the sequel and we give the definition of equality algebra and their proprieties. Hence we present the notions of filters in equality algebra. Moreover, the notion of congruence relation and some proprieties are given.

contents

1.1 Lattices

A partial order (order for short) is a binary relation $\leq$ over a set X which is reflexive ($a \leq a$ for any $a \in X$), antisymmetric ($a \leq b$ and $b \leq a$ implies $a = b$ for any $a, b \in X$) and transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$ for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset).

We shall be particularly interested in ordered sets $(L, \leq)$ in which $sup\{x, y\}$, $inf\{x, y\}$ exist for all $x, y \in L$.

Notation 1.1 *We shall adopt the following neater notation : we write $x \vee y$ (read as x **join** y) in place of $sup\{x, y\}$ when it exists and $x \wedge y$ (read as x **meet** y) in place of* $\inf\{x, y\}$ *when it exists.*

Definition 1.2 *Let $(L, \leq)$ be an ordered set.*

(i) *If $x \wedge y$ exist for all $x, y \in L$, then $(L, \leq)$ is called a meet-semilattice.*

Definition 1.3 *A meet-semilattice $(L, \wedge)$ is a non-empty set with a binary operations $\wedge$ satisfying the following properties :*

l1) *for every $a \in L$, $a = a \wedge a$,*

l2) *for every $a, b \in L$, $a \wedge b = b \wedge a$,*

l3) *for every $a, b, c \in L$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,*

Definition 1.4 *Let $(L, \leq)$ be an ordered set.*

(i) *If $x \vee y$, $x \wedge y$ exist for all $x, y \in L$, then $(L, \leq)$ is called a lattice.*

Now we recall the definition of lattices.

Definition 1.5 *A lattice* $(L, \vee, \wedge)$ *is a non-empty set with two binary operations* $\wedge$ *and* $\vee$ *satisfying the following properties :*

L1) *for every* $a \in L$, $a = a \vee a$ *and* $a = a \wedge a$,

L2) *for every* $a, b \in L$, $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$,

L3) *for every* $a, b, c \in L$, $(a \vee b) \vee c = a \vee (b \vee c)$ *and* $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,

L4) *for every* $a, b \in L$, $a = a \wedge (a \vee b)$ *and* $a = a \vee (a \wedge b)$.

Lemma 1.6 *Let* L *be a lattice and let* $a, b \in L$. *Then the following are equivalent :*

(i) $a \leq b$,

(ii) $a \vee b = b$,

(iii) $a \wedge b = a$.

Distributive lattices

Lemma 1.7 *Let* L *be a lattice. Then the following are equivalent:*

$(D1)$ $(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

$(D2)$ $(\forall a, b, c \in L)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 1.8 *Let* L *be a lattice.* L *is said to be distributive if it satisfies the distributive law,* $(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Filters and ideals

Definition 1.9 *A non-empty subset F of a lattice L is called a filter of L if for all $x, y \in L$*

1. *if $y \in F$ with $y \leq x$, then $x \in F$,*

2. *if $x, y \in F$ implies $x \wedge y \in F$.*

Definition 1.10 *A filter F of L is a prime filter if: $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in L$.*

Definition 1.11 *A non-empty subset I of a lattice L is called an ideal of L if for all $x, y \in L$*

1. *if $y \in I$ with $x \leq y$, then $x \in I$,*

2. *if $x, y \in I$ implies $x \vee y \in I$.*

Definition 1.12 *An ideal I of L is a prime ideal if: $x \wedge y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in L$.*

1.2 Generalities on equality algebra

Equality algebra

In this section, we recollect some definitions and results which will be used in this work and we shall not cite them every time they are used.

Definition 1.13 *[7]An algebra* $\varepsilon = (E, \wedge, \sim, 1)$ *of the type* $(2, 2, 0)$ *is called*

an equality algebra if it satisfies the following conditions, for all $x, y, z \in E$:

(E1) $(E, \wedge, 1)$ is a meet-semilattice with the top element 1,

(E2) $x \sim y = y \sim x$,

(E3) $x \sim x = 1$,

(E4) $x \sim 1 = x$,

(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,

(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,

(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is an equality operation.We write $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$. Also, two other operations are defined and called implication and equivalence operation, respectively:

$$x \to y = x \sim (x \wedge y) \tag{I}$$

$$x \longleftrightarrow y = (x \to y) \wedge (y \to x) \tag{II}$$

If $\sim$ coincides with $\longleftrightarrow$, then an equality algebra is called equivalential. An equality algebra $(E, \sim, \wedge, 1)$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E, we define the negation "$'$" on E by, $x' = x \to 0 = x \sim 0$, for all $x \in E$. If $(x')' = x$, for all $x \in E$, then the bounded equality algebra E is called involutive. An equality algebra E is called prelinear if 1 is the unique upper bound of the set $\{x \to y,\ y \to x\}$, for all $x, y \in E$. An equality algebra E is called commutative if $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in E$. A lattice equality algebra is an equality algebra which is a lattice, as well.

Example 1.14 *Let $\varepsilon = (E = \{0, a, b, c, 1\}, \wedge, \vee)$ be a lattice which defined by the tabeles*

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

.

and define the operation $\sim$ and $\to$ on E as follows

$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

.

Then by routine calculation, we can see that $(E, \wedge, \sim, 1)$ is prelinear equality algebra but is not commutative, because $(a \to 0) \to 0 = 1 \neq a = (0 \to a) \to a$.

The following propositions provide some properties of equality algebras.

Proposition 1.15 *[7]Let ε be an equality algebra and consider*

$$x \sim (x \wedge y \wedge z) \leq x \sim (x \wedge y)\,, \tag{E5a}$$

$$x \rightarrow (y \wedge z) \leq x \rightarrow y \tag{E5a'}$$

Then $(E5)$ is equivalent to $(E5a)$, which in turn is equivalent to $(E5a')$

Proof. Assuming $(E5)$, from $x \wedge y \wedge z \leq x \wedge y$, it follows that $x \sim (x \wedge y \wedge z) \leq x \sim (x \wedge y)$, so $(E5a)$ is obtained.

Assume $(E5a)$. Referring to $x \leq y \leq z$, $x \sim z = (x \wedge y \wedge z) \sim z \leq (z \wedge y) \sim z = y \sim z$ holds using $(E5a)$, and $x \sim z = (z \wedge x) \sim z \leq (x \wedge y \wedge z) \sim (z \wedge y) = x \sim y$ holds using $(E6)$. So $(E5)$ follows. By (I) we immediately see the equivalence of $(E5a)$ with $(E5a')$. ■

Corollary 1.16 *[7] In equality algebras we observe that $(E6)$ implies*

$$x \sim (x \wedge y) \leq (x \wedge z) \sim (x \wedge y \wedge z) \tag{E5b}$$

which is, by (I), equivalent to

$$x \rightarrow y \leq (x \wedge z) \rightarrow y \tag{E5b'}$$

Proposition 1.17 *[7]Let $E = (E, \wedge, \sim, 1)$ be an equality algebra. Then the following properties hold, for all $x, y, z \in E$*

(i) $x \sim y \leq x \leftrightarrow y \leq x \rightarrow y$,

(ii) $x \leq (x \sim y) \sim y$,

(iii) $x \sim y = 1$ *if and only if* $x = y$,

(iv) $x \rightarrow y = 1$ *and* $y \rightarrow x = 1$ *imply* $x = y$,

(v) $x \leq (x \rightarrow y) \rightarrow y$.

Proof. (i) Substitute $z = x$ into $(E6)$ to obtain $x \sim y \leq x \rightarrow y$. Substitute $z = y$ into $(E6)$ and use $(E2)$ to obtain $x \sim y \leq y \rightarrow x$. Hence, by (II), we obtain $x \sim y \leq x \leftrightarrow y$.

(ii) Using $(E4)$, $(E7)$, $(E2)$ and $(E4)$, respectively, we obtain $x = x \sim 1 \leq (x \sim y) \sim (1 \sim y) = (x \sim y) \sim (y \sim 1) = (x \sim y) \sim y$.

(iii) Assume $x \sim y = 1$, $x = x \sim 1 \leq (x \sim y) \sim (y \sim 1) \leq 1 \sim y \leq y$, it follows $x \leq y$. $y = y \sim 1 \leq (x \sim y) \sim (x \sim 1) \leq 1 \sim x \leq x$, it follows $y \leq x$, so $x = y$.

(iv) (I) and (iii) yield $x \rightarrow y = 1 \Leftrightarrow x \sim (x \wedge y) = 1 \Leftrightarrow x = x \wedge y \Leftrightarrow x \leq y$, in the same way we can show that $y \rightarrow x = 1 \Leftrightarrow y \leq x$.

(v) Using (ii), (I), (i), and $(E5a\prime)$, we get $x \leq (x \sim (x \wedge y)) \sim (x \wedge y) \leq (x \rightarrow y) \rightarrow (x \wedge y) \leq (x \rightarrow y) \rightarrow y$. ■

Proposition 1.18 *[1]Let* $E = (E, \wedge, \sim, 1)$ *be an equality algebra. Then the following properties hold, for all* $x, y, z \in E$

(i) $x \to y = 1$ *if and only if* $x \le y$*,*

(ii) $1 \to x = x$*,* $x \to 1 = 1$*,* $x \to x = 1$*,*

(iii) $x \le y \to x$*,*

(iv) $x \le (x \to y) \to y$*,*

(v) $x \to y \le (y \to z) \to (x \to z)$*,*

(vi) $x \le y \to z$ *if and only if* $y \le x \to z$*,*

(vii) $x \to (y \to z) = y \to (x \to z)$*.*

Proof. (i) By proposition 1.17 (i). $x \to y = 1$, $x \sim (x \wedge y) = 1$, it follows that $x = (x \wedge y)$, hence $x \le y$. if $x \le y$, then $x \to y = x \sim (x \wedge y) = x \sim x = 1$.

(ii) $1 \to x = 1 \sim (1 \wedge x) = 1 \sim x = x.x \to 1 = 1$, $x \to x = 1$, the same.

(iii) $x = 1 \to x \le (1 \wedge y) \to x$, (ii), $(E5b')$, follows that $x \le y \to x$.

(iv) By proposition 1.17 (ii) and (i) $x \le (x \sim y) \sim y \le (x \to y) \to y$. so $x \le (x \to y) \to y$.

(v) $x \to y = x \sim (x \wedge y) = (x \wedge y) \sim x \le ((x \wedge y) \sim (x \wedge y \wedge z) \sim (x \sim (x \wedge y \wedge z))$

$\le ((x \wedge y) \sim (x \wedge y \wedge z) \to (x \sim (x \wedge y \wedge z)) \le y \sim (y \wedge z) \to x \sim (x \wedge z) = y \to z \to x \to z$.

(vi) we have $x \le y \Rightarrow y \to z \le x \to z$. $y \to z = y \sim (y \wedge z) \le (y \wedge x) \sim (x \wedge y \wedge z) \le x \sim (x \wedge z) = x \to z$. $x \le y \to z$ it follows $(y \to z) \to z \le x \to z$, by (iv) $y \le (y \to z) \to z \le x \to z$, so $y \le x \to z$.

(vii) By (iv) we have $x \le (x \to z) \to z$ hence by $(E5b\prime)$ we get $x \to (y \to z) \ge ((x \to z) \to z) \to (y \to z)$. Because of (v) the right hand side is at least $y \to (x \to z)$. Thus we get $x \to (y \to z) \ge y \to (x \to z)$ and analogously we can show that$y \to (x \to z) \ge x \to (y \to z)$ too. ■

Proposition 1.19 *[1]Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. Then, for all $x, y, z \in E$, the following statements hold:*

(i) *$x \leq y$ implies $y \to z \leq x \to z$, $z \to x \leq z \to y$,*

(ii) *$x \to y = x \to (x \wedge y)$,*

(iii) *$x \to y \leq (z \to x) \to (z \to y)$,*

(iv) *$x \to y \leq (x \wedge z) \to (y \wedge z)$.*

Proof.

(i) $y \to z \leq (y \wedge x) \to z \leq x \to z$ $(E5b')$. $z \to y \geq z \to (y \wedge x) = z \to x$.

(ii) $x \to y = x \sim (x \wedge y) \leq x \to (x \wedge y)$. and $x \to (x \wedge y) \geq x \sim (x \wedge y) = x \to y$.

(iii) $x \to y = x \sim (x \wedge y) \leq x \to (x \wedge y) \leq (z \to x) \to (z \to (x \wedge y))$, $x \wedge y \leq y$. $x \to y \leq (z \to x) \to (z \to y)$.

(iv) $x \to y = x \sim (x \wedge y) \leq (x \wedge z) \sim (x \wedge y \wedge z) \leq (x \wedge z) \to (x \wedge y \wedge z) \leq (x \wedge z) \to (y \wedge z)$, by $(E5a)$, $(E5a')$.

■

Proposition 1.20 *[14] Let E be a lattice equality algebra. Then, for all $x, y \in E$, the following statements hold:*

(i) *$(x \vee y) \to z = (x \to z) \wedge (y \to z)$,*

(ii) *$x \to y = (x \vee y) \to y$.*

Proof. (i) Suppose that $x \vee y$ exists. Since $x \leq x \vee y$ and $y \leq x \vee y$, by Proposition 1.19 (i) $(x \vee y) \rightarrow z \leq x \rightarrow z$ and $(x \vee y) \rightarrow z \leq y \rightarrow z$. Hence, $(x \vee y) \rightarrow z$ is a lower bound in E of $\{x \rightarrow z,\ y \rightarrow z\}$. Let δ be an author lower bound for the set $\{x \rightarrow z,\ y \rightarrow z\}$. Then, $\delta \leq x \rightarrow z$ and $\delta \leq y \rightarrow z$ by Proposition 1.18 (vi) $x \leq \delta \rightarrow z$ and $y \leq \delta \rightarrow z$, then $x \vee y \leq \delta \rightarrow z$. Again by Proposition 1.18 (vi) $\delta \leq (x \vee y) \rightarrow z$. Then, $(x \vee y) \rightarrow z$ is the infimum in E for the set $\{x \rightarrow z,\ y \rightarrow z\}$. It follows that $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

(ii) By (i) $(x \vee y) \rightarrow y = (x \rightarrow y) \wedge (y \rightarrow y) = (x \rightarrow y) \wedge 1 = x \rightarrow y$. ■

Proposition 1.21 *[14] Let E be a bounded lattice equality algebra. Then, for all $x, y \in E$, the following statements hold:*

(i) $x \leq (x')'$,

(ii) $(x \vee y)' = x' \wedge y'$.

Proof.

(i) $x\prime = (x \rightarrow 0)\,,\ (x')' = (x \rightarrow 0) \rightarrow 0 \geq x$, from Proposition 1.18 (iv).

(ii) $(x \vee y)' = (x \vee y) \rightarrow 0 = (x \rightarrow 0) \wedge (y \rightarrow 0) = x' \wedge y'$, from Proposition 1.20 (i).

■

Theorem 1.22 *[14]Every commutative equality algebra is a lattice.*

Theorem 1.23 *[14]Any prelinear equality algebra is a distributive. lattice.*

1.3 Filters in equality algebra

Definition 1.24 *[8]Let $\varepsilon = (E, \sim, \wedge, 1)$ be an equality algebra and F be a non-empty subset of E. Then F is called a deductive system or a filter of ε if, for all $x, y \in E$, we have :*

(F1) $1 \in F$,

(F2) if $x \in F$ and $x \leq y$, then $y \in F$,

(F3) if $x \in F$ and $x \sim y \in F$, then $y \in F$.

Denote by $F(\varepsilon)$ the set of all filters of ε . Clearly, $F(\varepsilon\ )$ is closed under arbitrary intersections and $\{1\} \in F(\varepsilon\ )$, so $(F(\varepsilon), \subseteq)$ is a complete lattice. A filter F of ε is called a proper filter if $F \neq E$. It can be easily seen that, if ε is a bounded equality algebra, then a filter is proper if and only if it does not contain 0. A proper filter of ε is called a maximal filter if it is not properly contained in any other proper filter of ε.

Proposition 1.25 *[1]Let ε be an equality algebra. Then, $F \in F(\varepsilon)$ if and only if, for all $x, y \in E$,*

(i) $1 \in F$,

(ii) if x and $x \to y \in F$, then $y \in F$.

Proof. $(\Rightarrow)$ $(i)\, 1 \in F$. By $(F1)$.

(ii) $x, x \to y \in F$, imply $x, x \sim (x \wedge y) \in F$, implies by $(F3)$ $x \wedge y \in F$, and $x \wedge y \leq y$. so, by $(F2)$ $y \in F$.

$(\Leftarrow)\,(F1)$ $1 \in F$. By (i) .

$(F2)$ $x,\ x \leq y \in F$, imply $x \to y = x \sim (x \wedge y) = 1$, it follows that $x \to y \in F$. So $x, x \to y \in F$. By (ii) $y \in F$.

$(F3)$ $x,\ x \sim y \in F$, imply $x \in F$, and by $(E6)$, $x \sim y \leq (x \wedge x) \sim (x \wedge y) = x \to y \in F$, then $y \in F$. ∎

1.4 Filters and congruence relations on equality algebra

Definition 1.26 *[1]Let $\varepsilon = (E, \wedge, \sim, 1)$ be an equality algebra. A subset Θ of $E \times E$ is called a congruence relation on E, if it is an equivalence relation on E and, for all $x, y, z, w \in E$ such that $(x, z),\ (y, w) \in \Theta$, we have $(x \wedge y, z \wedge w) \in \Theta$ and $(x \sim y, z \sim w) \in \Theta$. Denote by $Con(\varepsilon)$ the set of all congruence relations on E.*

Proposition 1.27 *[1]If E is an equality algebra, $F \in F(\varepsilon)$ and the relations $\Theta_{\overrightarrow{F}}$ and Θ_F on E are defined by $(x, y) \in \Theta_{\overrightarrow{F}} \Leftrightarrow \{x \to y, y \to x\} \subseteq F$ and $(x, y) \in \Theta_F \Leftrightarrow x \sim y \in F$, then Θ_F, $\Theta_{\overrightarrow{F}} \in Con(\varepsilon)$, and $\Theta_{\overrightarrow{F}} = \Theta_F$.*

Proof. The reflexivity and the symmetry of Θ_F are trivially. For the transitivity of Θ_F let $x, y, z \in E$, such that $(x, y) \in \Theta_F$ and $(y, z) \in \Theta_F$, it follows that $x \sim y \in F$ and $y \sim z \in F$, since $x \sim y \le (y \sim z) \sim (x \sim z)$ and F is a filter, then $x \sim z \in F$. Thus $(x, z) \in \Theta_F$. Therefore Θ_F is an equivalence relation. For all $x, y, z, w \in E$ such that $(x, z),\ (y, w) \in \Theta_F$, we have $x \sim z \in F$ and $y \sim w \in F$, hence by $(E6)$ we have $(x \wedge y) \sim (z \wedge y) \in F$ and $(y \wedge z) \sim (w \wedge z) \in F$, it follows that $(x \wedge y, z \wedge w) \in \Theta$ and $(x \sim y, z \sim w) \in \Theta$ follows from $(E7)$. Therefore $\Theta_F \in Con(\varepsilon)$. The reflexivity and the symmetry of $\Theta_{\overrightarrow{F}}$ are trivially. For the transitivity of $\Theta_{\overrightarrow{F}}$ let $x, y, z \in E$, such that $(x, y) \in \Theta_{\overrightarrow{F}}$ and $(y, z) \in \Theta_{\overrightarrow{F}}$, it follows that $x \to y \in F$ and $y \to z \in F$, since $x \to y = x \sim (x \wedge y)$ and $y \to z = y \sim (y \wedge z) \le (x \wedge y) \sim (x \wedge y \wedge z)$ and F is a filter, then $x \sim (x \wedge y \wedge z) \in F$, hence $x \sim (x \wedge z) = x \to z \in F$. Similarly we can show that if $y \to x \in F$ and $z \to y \in F$, then $z \to x \in F$, Thus $(x, z) \in \Theta_{\overrightarrow{F}}$.Therefore $\Theta_{\overrightarrow{F}}$ is an equivalence relation. For all $x, y, z, w \in E$ such that $(x, z),\ (y, w) \in \Theta_{\overrightarrow{F}}$, we have $x \to z = x \sim (x \wedge z) \in F$ and $y \to w = y \sim (y \wedge w) \in F$, hence $(x \wedge y) \sim (x \wedge y \wedge z) \in F$ and $(x \wedge y \wedge z) \sim (x \wedge y \wedge z \wedge w) \in F$ (by $(E6)$ and) we have $(x \wedge y) \sim (x \wedge y \wedge z) \in F$ and $(x \wedge y \wedge z) \sim (w \wedge z) \in F$, it follows that $(x \wedge y, z \wedge w) \in \Theta_{\overrightarrow{F}}$. $x \to z = x \sim (x \wedge z) \le (x \sim y) \sim (y \sim (x \wedge z)) \in F$, and

$y \to w = y \sim (y \wedge w) \leq (y \sim (x \wedge z)) \sim ((x \wedge z) \sim (y \wedge w)) \in F$. It follows $(x \sim y) \sim ((x \wedge z) \sim (y \wedge w)) \in F$, from $(Ea5)$ it is easy to see that $(x \sim y, z \sim w) \in \Theta_{\overrightarrow{F}}$. Therefore, $\Theta_{\overrightarrow{F}} \in Con(\varepsilon)$. ■

Proposition 1.28 *[1]Let $E/F = \{[x]/x \in E\}$, where $[x] = \{y \in E/(x, y) \in \Theta_F\}$.The binary relation $\leq_F$ on E/F which is defined by $[x] \leq_F [y]$ if and only if $x \to y \in F$ is an order relation on E/F.*

Proof. For all $x \in E$, $x \to x = x \sim (x \wedge x) = x \sim x = 1 \in F$, hence $[x] \leq_F [x]$, so the relation $\leq_F$ is reflexive. $[x] \leq_F [y]$ and $[y] \leq_F [x]$, then $(x, y) \in \Theta_F$. so $[x] = [y]$.For the transitivity let $x, y, z \in E$, $[x] \leq_F [y]$ and $[y] \leq_F [z]$, $x \to y \in F$ and $y \to z \in F$, since $x \to y = x \sim (x \wedge y)$ and $y \to z = y \sim (y \wedge z) \leq (x \wedge y) \sim (x \wedge y \wedge z)$ and F is a filter, then $x \sim (x \wedge y \wedge z) \in F$, hence by $(E5a)$ $x \sim (x \wedge z) = x \to z \in F$, therefore $[x] \leq_F [z]$. ■

Theorem 1.29 *[1]Let ε be an equality algebra. Then there is a one-to-one correspondence between $F(\varepsilon)$ and $Con(\varepsilon)$.*

Proof. Define φ: $Con(\varepsilon) \to F(\varepsilon)$ by $\varphi(\Theta) = F_\Theta$ for all $\Theta \in Con(\varepsilon)$. Let $(x, y) \in \Theta$. Since $(y, y) \in \Theta$, it follows that $(x \sim y, y \sim y) \in \Theta$, so $(x \sim y, 1) \in \Theta$.It follows that $x \sim y \in F_\Theta$, hence $(x, y) \in \Theta_{F_\Theta}$, that is, $\Theta \subseteq \Theta_{F_\Theta}$.

Conversely, if $(x, y) \in \Theta_{F_\Theta}$, then $(x \sim y, 1) \in \Theta$, so $((x \sim y) \sim x, 1 \sim x) \in \Theta$, that is, $((x \sim y) \sim x, x) \in \Theta$. It follows that $(((x \sim y) \sim x) \wedge y, x \wedge y) \in \Theta$, that is, $(y, x \wedge y) \in \Theta$.Similarly $(x, x \wedge y) \in \Theta$, so $(x, y) \in \Theta$, that is, $\Theta_{F_\Theta} \subseteq \Theta$. Hence $\Theta = \Theta_{F_\Theta}$.

Let Θ_1, $\Theta_2 \in Con(\varepsilon)$ such that $\varphi(\Theta_1) = \varphi(\Theta_2)$. It follows that $\Theta_1 = \Theta_{F_{\Theta_1}} = \Theta_{F_{\Theta_2}} = \Theta_2$. Consider $F \in F(\varepsilon)$. We have : $x \in F$ if and only if $x \sim 1 \in F$ if and only if $(x, 1) \in \Theta_F$ if and only if $x \in \varepsilon_{\Theta_F}$, hence $F = \varphi(\Theta_F)$. We conclude that φ is a one-to-one correspondence between $F(\varepsilon)$ and $Con(\varepsilon)$. ■

Theorem 1.30 *[3]Let $(E, \wedge, \sim, 1)$ be an equality algebra and $F \in F(\varepsilon)$. Then $(E/F, \sim_F, \wedge_F, 1_F)$ is an equality algebra, where for every $x, y \in E$, $1_F = [1]$, $[x] \sim_F [y] = [x \sim y]$, $[x] \wedge_F [y] = [x \wedge y]$.*

Theorem 1.31 *[1]An equality algebra* $(E, \wedge, \sim, 1)$ *is equvalential if and only if for all* $x, y \in E$, $x \sim y = (x \sim (x \wedge y)) \wedge (y \sim (x \wedge y))$.

Proof. $(\Rightarrow)$ equvalential ($\sim$ conncide with $\longleftrightarrow$) from definition $x \longleftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x) = (x \sim (x \wedge y)) \wedge (y \sim (x \wedge y)) = x \sim y$.

$(\Leftarrow)$ $x \sim y = (x \sim (x \wedge y)) \wedge (y \sim (x \wedge y)) = (x \rightarrow y) \wedge (y \rightarrow x) = x \longleftrightarrow y$. $\blacksquare$

Chapter 2

(Positive) Implicative filters in equality algebra

Abstract

In this Chapter, we present the notions of positive implicative filters and implicative filters in equality algebra. And some proprieties of them are given.

contents

2.1 Positive implicative filters

In this section, we give some characterizations of positive implicative filters

and give the relaions between them.

Definition 2.1 *[1]Let F be a non-empty subset of E such that $1 \in F$. Then F is called a positive implicative filter if $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$, for all $x, y, z \in E$.*

The following examples give an example of positive implicative filters in equality algebras.

Example 2.2 *Let $(E = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\sim$ and $\to$ on E by*

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

By calculations, we can see that $(E, \wedge, \sim, 1)$ is an equality algebra and $F = \{1, b\}$ is a positive implicative filter of E.

Lemma 2.3 *[1]Any positive implicative filter of E is a filter.*

Proof. Let x, $x \to y \in F$. Then $1 \to (x \to y) = x \to y \in F$ and $1 \to x = x \in F$. Since F is a positive implicative filter, $y = 1 \to y \in F$. Hence, F is a filter of E. ■

Remark 2.4 *In the following Example we show that a filter in equality algebra is not necessarily a positive implicative algebra.*

Example 2.5 *Let $(E = \{0, a, b, c, d, 1\}, \wedge, \vee)$ be a lattice defined by the following tables*

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	d	0	d	d	d
1	0	a	b	c	d	1

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	a	1	a	1
b	b	a	b	1	a	1
c	c	1	1	c	c	1
d	d	a	a	c	d	1
1	1	1	1	1	1	1

and define the operations $\sim$ and $\rightarrow$ on E by

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	0	a	b	c	d	1
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then, by calculations, we can see that $(E, \wedge, \sim, 0, 1)$ is an equality algebra and $F = \{1, c\}$ is a filter, but it is not a positive implicative filter. Because $a \rightarrow (a \rightarrow 0) = c \in F$ and $a \rightarrow a = 1 \in F$, but $a \rightarrow 0 = d \notin F$.

Proposition 2.6 *[1]Let F be a non-empty subset of E. Then, for all $x, y, z \in E$, the following statements are equivalent:*

(i) F is a positive implicative filter of E,

(ii) F is a filter and if $x \rightarrow (x \rightarrow y) \in F$, then $x \rightarrow y \in F$,

(iii) F is a filter and if $z \rightarrow (y \rightarrow x) \in F$, then $(z \rightarrow y) \rightarrow (z \rightarrow x) \in F$,

(iv) $1 \in F$ *and if* $z \in F$ *and* $z \to (x \to (x \to y)) \in F$*, then* $x \to y \in F$*.*

Proof. *(i)* $\Rightarrow$ *(ii)* Let F be a positive implicative filter. Then, by Lemma 2.3, F is a filter of E. If $x \to (x \to y) \in F$, since $x \to x = 1 \in F$ and F is a positive implicative filter, $x \to y \in F$.

(ii) $\Rightarrow$ *(iii)* Let $z \to (y \to x) \in F$. By Propositions 1.18 *(vii)*, 1.19 *(i)* and *(iii)*, $z \to (z \to ((z \to y) \to x)) = z \to ((z \to y) \to (z \to x)) \geq z \to (y \to x)$. Since F is a filter and $z \to (y \to x) \in F$, we get $z \to (z \to ((z \to y) \to x)) \in F$. Then, by assumption $z \to ((z \to y) \to x) \in F$. Thus, by Proposition 1.18 *(vii)*, $(z \to y) \to (z \to x) \in F$.

(iii) $\Rightarrow$ *(iv)* Since F is a filter, $1 \in F$. If $z \in F$ and $z \to (x \to (x \to y)) \in F$, then $x \to (x \to y) \in F$. By assumption, $(x \to x) \to (x \to y) \in F$, and so $x \to y \in F$.

(iv) $\Rightarrow$ *(i)* Let $z \to (y \to x) \in F$ and $z \to y \in F$. By Proposition 1.19 *(iii)*, $z \to (y \to x) \leq (z \to y) \to (z \to (z \to x))$.Since F is a filter and $z \to (y \to x) \in F$, we have $(z \to y) \to (z \to (z \to x)) \in F$. Then, by *(iv)*, $z \to x \in F$. ■

Lemma 2.7 *[1]Let F be a filter of E. Then the following properties hold:*

(i) $x \sim y \in F$ *and* $y \sim z \in F$ *imply* $x \sim z \in F$*,*

(ii) $x \to y \in F$ *and* $y \to z \in F$ *imply* $x \to z \in F$*.*

Proof. *(i)* If $x \sim y \in F$ and $y \sim z \in F$, then by $(E7)$ and $(E2)$, $x \sim y \leq (y \sim z) \sim (x \sim z)$. Since F is a filter, $(y \sim z) \sim (x \sim z) \in F$, and so $x \sim z \in F$.

(ii) If $x \to y \in F$ and $y \to z \in F$, then by Proposition 1.19 *(iii)*, $y \to z \leq (x \to y) \to (x \to z)$. Since F is a filter, $(x \to y) \to (x \to z) \in F$, and so $x \to z \in F$. ■

Proposition 2.8 *[1]Let F be a filter of E. Then F is a positive implicative filter if and only if, for all $x, y \in E, (x \wedge (x \to y)) \to y \in F$*

Proof. $(\Rightarrow)$ Since $x \wedge (x \to y) \leq x \to y$ and $x \wedge (x \to y) \leq x$, we have $(x \wedge (x \to y)) \to (x \to y) = 1 \in F$ and $(x \wedge (x \to y)) \to x = 1 \in F$. Since F is a positive implicative filter, $(x \wedge (x \to y)) \to y \in F$.

($\Leftarrow$) Let $x \to (y \to z) \in F$ and $x \to y \in F$. By Proposition 1.19 (iv), $x \to (y \to z) \leq (x \wedge y) \to (y \wedge (y \to z))$. Since F is a filter, $(x \wedge y) \to (y \wedge (y \to z)) \in F$. By Proposition 1.19 (ii), $x \to y \leq x \to (x \wedge y) \in F$, and so by Lemma 2.7 (ii), $x \to (y \wedge (y \to z)) \in F$. By assumption, $(y \wedge (y \to z)) \to z \in F$. Thus, by Lemma 2.7 (ii), $x \to z \in F$. ■

Corollary 2.9 *[1]Suppose F and G are two filters of E and $F \subseteq G$. If F is a positive implicative filter, then G is a positive implicative filter, too.*

Proof. By Proposition 2.8, for all $x, y \in F$, $[x \wedge (x \to y)] \to y \in F$ and since $F \in G$, then $[x \wedge (x \to y)] \to y \in G$. ■

Corollary 2.10 *[1]Every filter of E is a positive implicative filter if and only if $\{1\}$ is a positive implicative filter.*

Proof. By Corollary 2.9 the proof is clear. ■

Proposition 2.11 *[1]Let F be a filter of E. Then F is a positive implicative filter if and only if every filter of equality algebra E/F is a positive implicative filter.*

Proof. Let F be a filter of E. Then by Proposition 2.8, F is a positive implicative filter of E if and only if for any $x, y \in E$, $(x \wedge (x \to y)) \to y \in F$, if and only if $[(x \wedge (x \to y)) \to y] = [1]$ if and only if, for every $x, y \in E$, $([x] \wedge_F ([x] \to_F [y])) \to_F [y] = [1]$ if and only if, by Proposition 2.8, $\{[1]\}$ is a positive implicative filter of E/F if and only if, by Corollary 2.10, every filter of E/F is a positive implicative filter. ■

Proposition 2.12 *[1]Let F be a non-empty subset of E. Then F is a positive implicative filter if and only if, for any $a \in E$, $F_a = \{x \in E / a \to x \in F\}$ is the least filter of E containing F and $\{a\}$.*

Proof. ($\Rightarrow$) Suppose that F is a positive implicative filter of E and $a \in E$. Since $a \to 1 = 1 \in F$, $1 \in F_a$. If x, $x \to y \in F_a$, then $a \to x \in F$ and $a \to (x \to y) \in F$. Since F is a positive implicative filter, we have $a \to y \in F$, and so $y \in F_a$. Therefore,

F is a filter of E. Now, let $x \in F$. By Proposition 1.18 (iii), $x \leq a \to x$ and $x \in F$, then $a \to x \in F$, and so, $x \in F_a$. Hence, $F \subseteq F_a$. Moreover, $a \to a = 1 \in F$, then $a \in F_a$. Thus, $F \cup \{a\} \subseteq F_a$. Let G be a filter of E such that $F \cup \{a\} \subseteq G \subseteq F_a$. Then $a \to x \in F \subseteq G$, for every $x \in F_a$. Since G is a filter and $a \in G$, $x \in G$. Thus $F_a \subseteq G$, and so $F_a = G$. Hence, F_a is the least filter of E containing F and $\{a\}$.

($\Leftarrow$) Let $x, y \in E$. If $x \to (x \to y) \in F$, then $x \to y \in F_x$. Since F_x is a filter containing $\{x\}$, we have $y \in F_x$, and so $x \to y \in F$. Thus, by Proposition 2.6 (ii), F is a positive implicative filter. ■

2.2 Implicative filters

Definition 2.13 *[1]Let F be a non-empty subset of E. Then F is called an implicative filter if $1 \in F$ and if $z \to ((x \to y) \to x) \in F$ and $z \in F$, then $x \in F$, for all $x, y, z \in E$.*

The following example shows that implicative filter in equality algebras
exists.

Example 2.14 *Let $(E = \{0, a, b, c, 1\}, \leq)$ be a chain whre $0 \leq a \leq b \leq c \leq 1$. Define the operations $\sim$ and $\to$ on E by*

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	a
b	0	b	1	c	b
c	0	b	c	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	a	b	c	1

Then, by routine calculations we can see that $(E, \sim, \wedge, 1)$ is an equality algebra and $F = \{1, a, b, c\}$ is an implicative filter of E.

Lemma 2.15 *[1]Any implicative filter of E is a filter.*

Proof. Let x, $x \to y \in F$, then $x \to y = x \to ((y \to 1) \to y) \in F$. Since F is an implicative filter, $y \in F$. Hence, F is a filter. ■

Example 2.16 *In Example 2.2, $F = \{1, b\}$ is a filter which is not an implicative filter. Because $1 \to ((a \to 0) \to a) \in F$, but $a \notin F$.*

Proposition 2.17 *[1]Let F be a filter of bounded equality algebra E. For all $x, y, z \in E$, the following statements are equivalent:*

(i) *F is an implicative filter of E,*

(ii) *$(x \to y) \to x \in F$ implies $x \in F$,*

(iii) *$x' \to x \in F$ implies $x \in F$,*

(iv) *$x \to (z' \to y) \in F$ and $y \to z \in F$ imply $x \to z \in F$,*

(v) *$x \to (y' \to y) \in F$ implies $x \to y \in F$.*

Proof. $(i) \Rightarrow (ii)$ Let $(x \to y) \to x \in F$. Then, by Proposition 1.18 (ii), $(x \to y) \to x = 1 \to ((x \to y) \to x) \in F$ and $1 \in F$. Since F is an implicative filter, $x \in F$.

$(ii) \Rightarrow (i)$ Suppose $z \in F$ and $z \to ((x \to y) \to x) \in F$. Since F is a filter of E, $(x \to y) \to x \in F$ and by (ii), $x \in F$.

$(ii) \Rightarrow (iii)$ If $x' \to x \in F$, then $x' \to x = (x \to 0) \to x \in F$ and by (ii), $x \in F$.

$(iii) \Rightarrow (ii)$ Let $(x \to y) \to x \in F$. Since $0 \leq y$, by Proposition 1.19 (i), $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. Since F is a filter of E, $x' \to x \in F$, and so by (iii), $x \in F$.

$(iii) \Rightarrow (iv)$ Let $x \to (z' \to y) \in F$ and $y \to z \in F$. By Proposition 1.18 (vii) and (vi), $x \to (z' \to y) = z' \to (x \to y) \leq ((x \to y) \to (x \to z)) \to (z' \to (x \to z))$. Since F is a filter, $((x \to y) \to (x \to z)) \to (z' \to (x \to z)) \in F$. Since $y \to z \leq (x \to y) \to (x \to z)$ and $y \to z \in F$, $(x \to y) \to (x \to z) \in F$. Thus, by Proposition 1.25, $z' \to (x \to z) \in F$. Since $z' \to (x \to z) \leq (x \to z)' \to (x \to z), (x \to z)' \to (x \to z) \in F$, and so by (iii), $x \to z \in F$. $(iv) \Rightarrow (v)$ If $x \to (y\prime \to y) \in F$, then it is enough to choose $z = y$ in (iv).Since$y \to y = 1 \in F$,$x \to y \in F$.

$(v) \Rightarrow (iii)$ If $x' \to x \in F$, then $1 \to (x' \to x) \in F$. Thus, by (v), $x = 1 \to x \in F$. ■

Proposition 2.18 *[1]Let F be a filter of bounded equality algebra E. Then F is an implicative filter if and only if $(x' \to x) \to x \in F$, for any $x \in E$.*

Proof. Suppose that F is an implicative filter. Let $\alpha = (x' \to x) \to x$, for $x \in F$. Then

$$\begin{aligned}
\alpha' \to \alpha &= (\alpha \to 0) \to \alpha \\
&= (((x' \to x) \to x) \to 0) \to ((x' \to x) \to x), && \text{by Proposition 1.18 } (vii) \\
&= (x' \to x) \to ((((x' \to x) \to x) \to 0) \to x), && \text{by Proposition 1.18 } (v) \\
&\geq (((x' \to x) \to x) \to 0) \to x' \\
&= (((x' \to x) \to x) \to 0) \to (x \to 0), && \text{by Proposition 1.18 } (v) \\
&\geq x \to ((x' \to x) \to x), && \text{by Proposition 1.18 } (vii) \\
&= (x' \to x) \to (x \to x) \\
&= (x' \to x) \to 1 \\
&= 1.
\end{aligned}$$

Hence, $\alpha' \to \alpha \in F$, and so, by Proposition 2.17 (iii), $\alpha \in F$.

Conversely, suppose $(x \to y) \to x \in F$. Since $0 \leq y$, by Proposition 1.19 (i), $x \to 0 \leq x \to y$. Thus, $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. Since $(x \to y) \to x \in F$ and F is a filter of E, $x' \to x \in F$. Also, by assumption, $(x' \to x) \to x \in F$. Then, $x \in F$. Thus, by Proposition 2.17 (ii), F is an implicative filter. ■

In the following, we investigate the relation between positive and implicative filters.

Theorem 2.19 *[1]Any implicative filter of E is a positive implicative filter.*

Proof. By Proposition 2.8, it is enough to prove that $(x \wedge (x \to y)) \to y \in F$, for all $x, y \in E$. For this, $x \wedge (x \to y) \leq x$, $x \wedge (x \to y) \leq x \to y$ and, by Proposition 1.19 (i), $x \wedge (x \to y) \leq x \to y \leq (x \wedge (x \to y)) \to y$. Thus, $((x \wedge (x \to y)) \to y) \to y \leq (x \wedge (x \to y)) \to y$, and so $(((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y) = 1 \in F$. Since F is an implicative filter, by Proposition 2.17 (ii), $(x \wedge (x \to y)) \to y \in F$. ■

The following example shows that not every positive implicative filter of E is an implicative.

Example 2.20 *In Example 2.2, $F = \{1, b\}$is a positive implicative filter which is not an implicative filter. Because, $(a \to 0) \to a = 1 \in F$, but $a' \notin F$.*

Theorem 2.21 *[1]Let F be a positive implicative filter of bounded equality algebra E. Then F is an implicative filter if and only if $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$.*

Proof. Let F be an implicative filter of E and $(x \to y) \to y \in F$, for any $x, y \in E$. Since $x \leq (y \to x) \to x$, by Proposition 1.19 (i), $((y \to x) \to x)' \leq x' = x \to 0 \leq x \to y$. Then, by Proposition 1.19 (i), $(x \to y) \to y \leq ((y \to x) \to x)' \to y$. Since $y \leq (y \to x) \to x$, $((y \to x) \to x)' \to y \leq ((y \to x) \to x)' \to ((y \to x) \to x)$. Thus, $(x \to y) \to y \leq ((y \to x) \to x)' \to ((y \to x) \to x)$. By Lemma 2.15, F is a filter, then $((y \to x) \to x)' \to ((y \to x) \to x) \in F$. Hence, by Proposition 2.17 (iii), $(y \to x) \to x \in F$.

Conversely. Let F be a positive implicative filter and $x' \to x \in F$. By Propositions 1.21 (i) and 1.19 (i), $x' \to x \leq x' \to x''$. Since F is a filter, $x' \to x'' \in F$. By Proposition 2.6 (ii), $x' \to 0 \in F$. By assumption, we have $(0 \to x) \to x = x \in F$. Hence, by Proposition 2.17 (iii), F is an implicative filter. ■

Corollary 2.22 *[1]Let F be a positive implicative filter of bounded equality algebra E. Then F is an implicative filter if and only if $x'' \in F$ implies $x \in F$.*

Corollary 2.23 *[1]In every involutive equality algebra, implicative filters and positive implicative filters coincide.*

Lemma 2.24 *[1]If E is a bounded lattice equality algebra and for every $x \in E$, $x \vee x' = 1$, then $x \wedge x' = 0$.*

Proof. Let $x \vee x' = 1$. Then, by Proposition 1.21 (i) and (ii),

$x' \wedge x \leq x' \wedge x'' = (x \vee x')' = 1' = 0$. Thus,$x \wedge x' = 0$. ■

Notation 2.25 *If E is involutive, then the converse of Lemma 2.24 holds.*

Let$x \wedge x' = 0$. Then, by Proposition 1.21 (ii), $x \vee x' = (x \vee x')'' = (x' \wedge x'')' = (x' \wedge x)' = 0' = 1$.

Proposition 2.26 *[1]If F is an implicative filter of E, then every filter G of E which contains F is an implicative filter.*

Proof. Let F be an implicative filter. Then, by Theorem 2.19, F is a positive implicative filter. Thus, by Corollary 2.9, G is a positive implicative filter. Suppose $(x \to y) \to y \in G$. By Theorem 2.21, it is enough to prove that

$(y \to x) \to x \in G$. For this, let$u = (x \to y) \to y$. Since $u \to ((x \to y) \to y) = 1 \in F$ and F is a positive implicative filter, by Proposition 2.6 (iii), $(u \to (x \to y)) \to (u \to y) \in F$. Then, by Proposition 1.18 (vii), $(x \to (u \to y)) \to (u \to y) \in F$. Since$F$ is an implicative filter, by Theorem 2.21, $((u \to y) \to x) \to x \in F$. Thus, $((u \to y) \to x) \to x \in G$. By Proposition 1.18 (iv) and (v), $(x \to y) \to y \leq (((x \to y) \to y) \to y) \to y = (u \to y) \to y \leq$

$(y \to x) \to ((u \to y) \to x) \leq (((u \to y) \to x) \to x) \to ((y \to x) \to x)$.

By assumption,G is a filter, and so $(((u \to y) \to x) \to x) \to ((y \to x) \to x) \in G$. Since $((u \to y) \to x) \to x \in G$, we have $(y \to x) \to x \in G$. Hence, G is an implicative filter. ■

Proposition 2.27 *[1]In any bounded equality algebra E, the following conditions are equivalent:*

(i) $\{1\}$ *is an implicative filter,*

(ii) *every filter of E is an implicative filter,*

(iii) $F(a) = \{x \in E | x \geq a\}$ *is an implicative filter, for any $a \in E$,*

(iv) $(x \to y) \to x = x$, *for all $x, y \in E$,*

(v) $x \vee x\prime = 1$,

(vi) $x\prime \to x = x$,

Proof. $(i) \Rightarrow (ii)$ By Proposition 2.26, the proof is clear.

$(ii) \Rightarrow (iii)$ Since $\{1\}$ is an implicative filter, by Theorem 2.19, $\{1\}$ is a positive implicative filter. Since, for any $a \in E$, $1 \geq a$, we get $1 \in F\ (a)$. Let $x, x \to y \in F(a)$. Then $a \to x = 1$ and $a \to (x \to y) = 1$. Thus,$a \to y = 1$, and so $y \in F(a)$. Hence, $F(a)$ is a filter. By (ii), $F(a)$ is an implicative filter.

$(iii) \Rightarrow (iv)$ Since $(x \to y) \to x \in F((x \to y) \to x)$, by,$(iii)$ $F((x \to y) \to x)$ is an implicative filter. Then, by Proposition 2.17 (ii),$x \in F((x \to y) \to x)$, and so $x \geq (x \to y) \to x$.Also, we have $x \leq (x \to y) \to x$. Hence, $x = (x \to y) \to x$.

$(iv) \Rightarrow (v)$ At first, we prove that E is a commutative equality algebra. For this, let $x, y \in E$. Then, by (iv) and Proposition 1.18 (v), $(y \to x) \to x = (y \to x) \to ((x \to y) \to x) \geq (x \to y) \to y$. By a similar way, $(x \to y) \to y \geq (y \to x) \to x$. Thus, E is commutative, and so, by Theorem 1.22, E is a lattice such that $x \vee y = (x \to y) \to y$. Hence, by (iv), $x \vee x' = (x' \to x) \to x = ((x \to 0) \to x) \to x = x \to x = 1$.

$(v) \Rightarrow (vi)$ By Proposition 1.20 (i) and (v), $x' \to x = (x \vee x') \to x = 1 \to x = x$.

$(vi) \Rightarrow (iv)$ Since $x' \leq x \to y$, by Proposition 1.19 (i), $x' \to x \geq (x \to y) \to x$. Then, by $(vi), x \geq (x \to y) \to x$. By Proposition 1.18 (iii), $x \leq (x \to y) \to x$, and so $x = (x \to y) \to x$.

$(v) \Rightarrow (i)$ Assume that (v) holds, then the condition (vi) holds. Thus, by Proposition 2.18,$\{1\}$ is an implicative filter. ■

Theorem 2.28 *[1]Let F be a filter of E. Then the following conditions are equivalent:*

(i) *F is maximal and implicative filter,*

(ii) *F is maximal and positive implicative filter,*

(iii) *$x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for all $x, y \in E$.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 2.19, the proof is clear.

$(ii) \Rightarrow (iii)$ Suppose F is a positive implicative filter and $x, y \notin F$. By Proposition 2.12, F_y is the least filter containing F and y, that is, $F \subsetneqq Fy \subseteq E$. Since F is a maximal filter, $Fy = E$. Then $x \in F_y$, and so$y \to x \in F$. By a similar way, since $x \notin F, x \to y \in F$.

$(iii) \Rightarrow (i)$ Suppose F is not an implicative filter. Then, by Proposition 2.17 (ii), there exist $x, y \in E$ such that $(x \to y) \to x \in F$, and $x \notin F$. If $y \in F$, since $y \leq x \to y$ and F is a filter, $x \to y \in F$. Thus, $x \in F$, which is a contradiction. If $y \notin F$, then by (iii), $x \to y \in F$, since F is a filter and $(x \to y) \to x \in F, x \in F$, which is a contradiction. Hence, F is an implicative filter. Now, we prove that F is a maximal filter. Let G be a filter of E such that $F \subsetneq G \subseteq E$ and $a \in G \backslash F$. Since F_a is the least filter containing F and a, we have $F \subseteq F_a \subseteq G \subseteq E$. Let $u \in E$. If $u \in F$, then $u \in F_a$. If $u \notin F$, since $a \notin F$, then $a \to u \in F$. By Proposition 2.12, $u \in F_a$. Thus, $F_a = E$, and so $G = E$. Hence, F is a maximal filter. ∎

Corollary 2.29 *[1]Let F be a maximal filter of E. Then F is an implicative filter if and only if F is a positive implicative filter*

Chapter 3

Fantastic and Boolean filters in equality algebra

Abstract

In this Chapter, we give the notions of fantastic and Boolean filters in equality algebra. Hence we present some properties of them. Moreover, the notion of prime filter in equality algebra is given.

contents

3.1 Fantastic filters in equality algebra

In this section, we introduce the notion of fantastic filters of equality algebras and give some properties of them.

Definition 3.1 *[1]A non-empty subset F of E is called a fantastic filter if*

(i) $1 \in F$,

(ii) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$, for all $x, y, z \in E$.

Example 3.2 *Let* $(E = \{0, a, b, c, 1\}, \wedge, \vee)$ *be a lattice with the following*

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

.

Define the operations $\sim$ *and* $\to$ *on* E *by*

$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then, by routine calculations, we can see that $(E, \wedge, \sim, 1)$ *is an equality algebra and* $\{1, a, b, c\}$*is a fantastic filter.*

Lemma 3.3 *Any fantastic filter of* E *is a filter.*

Proof. Let z, $z \to x \in F$. Since $z \to (1 \to x) = z \to x \in F$ and F is a fantastic filter, we have $((x \to 1) \to 1) \to x = x \in F$. Hence, F is a filter. ■

Example 3.4 *Let E be an equality algebra as Example 3.2. By routine calculations, we can see that $F = \{1, a\}$ is a filter which is not a fantastic filter. Because, $0 \to b = 1 \in F$, but $((b \to 0) \to 0) \to b = b \notin F$.*

Proposition 3.5 *[1]Let F be a filter of E. Then the following conditions are equivalent:*

(i) *F is a fantastic filter of E,*

(ii) *$y \to x \in F$ implies $((x \to y) \to y) \to x \in F$, for all $x, y \in E$,*

(iii) *if E is a lattice, then $((x \to y) \to y) \to ((y \to x) \to x) \in F$, for all $x, y \in E$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that F a fantastic filter and $y \to x \in F$. Let $z = 1$. Since $1 \to (y \to x) = y \to x \in F$ and $1 \in F, ((x \to y) \to y) \to x \in F$.

$(ii) \Rightarrow (i)$ Since F is a filter, if $z \to (y \to x) \in F$ and $z \in F$, then $y \to x \in F$. Thus by $(ii), ((x \to y) \to y) \to x \in F$.

$(ii) \Rightarrow (iii)$ Let E be a lattice. By Proposition 1.20 (i), $(x \vee y) \to y = x \to y$ and $y \to (x \vee y) = 1 \in F$, for any $x, y \in E$. Then, by $(ii), (((x \vee y) \to y) \to y) \to (x \vee y) \in F$, and so $((x \to y) \to y) \to (x \vee y) \in F$. Moreover, by Proposition 1.18 (iii) and (iv), $x \leq (y \to x) \to x$ and $y \leq (y \to x) \to x$. Thus, $x \vee y \leq (y \to x) \to x$, and so $(x \vee y) \to ((y \to x) \to x) = 1 \in F$. Then, by Lemma 2.7 (ii), $((x \to y) \to y) \to ((y \to x) \to x) \in F$.

$(iii) \Rightarrow (ii)$ Let $x, y \in E$ and $y \to x \in F$. Since $(y \to x) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((y \to x) \to x) \in F$ and F is a filter, we have $((x \to y) \to y) \to x \in F$. ■

Corollary 3.6 *[1]In a commutative equality algebra, any filter is a fantastic filter.*

Proof. By Theorem 1.22 and Proposition 3.5 (iii), the proof is clear. ■

Proposition 3.7 *[1]Suppose F and G are two filters of E and $F \subseteq G$. If F is a fantastic filter, then so is G.*

Proof. Let $y \to x \in G$, for any $x, y \in E$. By Proposition 1.18 (vii), $y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F$. Since F is a fantastic filter, $((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x) \in F \subseteq G$. Thus, by Proposition 1.18 (vii), $(y \to x) \to (((((y \to x) \to x) \to y) \to y) \to x) \in G$.Since G is a filter and $y \to x \in G$, $((((y \to x) \to x) \to y) \to y) \to x \in G$. By Proposition 1.18 (v) and (vii), $((((((y \to x) \to x) \to y) \to y) \to x)) \to (((x \to y) \to y) \to x) \geq x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = 1 \in G$. Since G is a filter, $((((((y \to x) \to x) \to y) \to y) \to x)) \to (((x \to y) \to y) \to x) \in G$, and so $((x \to y) \to y) \to x \in G$. Hence, G is a fantastic filter ■

Corollary 3.8 *[1]Every filter of E is a fantastic filter if and only if $\{1\}$ is a fantastic filter.*

Proposition 3.9 *[1]An equality algebra E is commutative if and only if $\{1\}$ is a fantastic filter.*

Proof. $(\Rightarrow)$ Suppose E is a commutative equality algebra. Since $\{1\}$ is a filter, by Corollary 3.6, it is a fantastic filter.

$(\Leftarrow)$ Let $\alpha = (y \to x) \to x$, for $x, y \in E$. Then $y \to \alpha = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1$. Since $\{1\}$is a fantastic filter, $((\alpha \to y) \to y) \to \alpha \in \{1\}$. Since $x \leq \alpha$, by Proposition 1.19 (i), $((\alpha \to y) \to y) \to \alpha \leq ((x \to y) \to y) \to \alpha$. Then $((x \to y) \to y) \to \alpha = 1$, and so $(x \to y) \to y \leq (y \to x) \to x$. By a similar way, $(y \to x) \to x \leq (x \to y) \to y$. Hence, E is commutative. ■

Proposition 3.10 *[1]Let F be a filter of E. Then F is a fantastic filter if and only if every filter of E/F is a fantastic filter.*

Proof. $(\Rightarrow)$ Suppose F is a fantastic filter of E and $x, y \in E$ such that $[x] \to_F [y] = [1]$.Then, x$\to y \in F$. Thus, $((y \to x) \to x) \to y \in F$, and so $(([y] \to_F [x]) \to_F [x]) \to_F$

$[y] = [((y \to x) \to x) \to y] = [1]$, which proves that $\{[1]\}$ is a fantastic filter of E/F . By Corollary 3.8, every filter of E/F is a fantastic filter.

($\Leftarrow$) Let $x, y \in E$ such that $x \to y \in F$. Then $[x] \to_F [y] = [1]$. Since $\{[1]\}$ is a fantastic filter of E/F, we have $[((y \to x) \to x) \to y] = [1]$, and so $((y \to x) \to x) \to y \in F$. Thus, F is a fantastic filter of E. ■

Corollary 3.11 *[1]F is a fantastic filter of E if and only if E/F is a commutative equality algebra.*

Proof. Let F be a filter of E. By Proposition 3.10, Corollary 3.8, and Proposition 3.9, F is a fantastic filter of E if and only if every filter of E/F is a fantastic filter if and only if $\{[1]\}$is a fantastic filter if and only if E/F is a commutative equality algebra. ■

Theorem 3.12 *[1]Any implicative filter of E is a fantastic filter.*

Proof. Let F be an implicative filter of E and $y \to x \in F$. Since $x \leq ((x \to y) \to y) \to x$, by Proposition 1.19 (i), $(((x \to y) \to y) \to x) \to y \leq x \to y$. Then, by Proposition 1.19 (i), $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x)$, by Proposition 1.18 (vii)$= ((x \to y) \to y) \to ((x \to y) \to x)$, by Proposition 1.19 (iii) $\geq y \to x$ By Lemma 3.3, F is a filter, and so, $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in F$ and, by Proposition 2.17 (ii), $((x \to y) \to y) \to x \in F$. ■

Example 3.13 *Let $(E = \{0, a, b, 1\}, \leq)$ be a chain whre $0 \leq a \leq b \leq 1$. Define the operations $\sim$ and $\to$ on E by*

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	1	1	1
1	0	a	b	1

By routine calculations, we can see that $(E, \sim, \wedge, 0, 1)$ is an equality algebra and $F = \{1, b\}$ is a fantastic filter, which is not an implicative filter. Because, $(a \to 0) \to a = 1 \in F$, but $a \notin F$

Notation 3.14 *In Example 2.2, $F = \{1, b\}$ is a positive implicative filter, but it is not a fantastic filter. Because $0 \to a = 1 \in F$, but $((a \to 0) \to 0) \to a = a \notin F$. Moreover, in Example 2.5, $G = \{1, c\}$ is a fantastic filter, but it is not a positive implicative filter. Because $a \to (a \to b) = 1 \in G$, but $a \to b = a \notin G$. This shows that positive implicative and fantastic filters do not coincide, in general.*

Theorem 3.15 *[1]F is an implicative filter of E if and only if F is a positive implicative filter and fantastic filter.*

Proof. $(\Rightarrow)$ By Theorems 2.19 and 3.12, the proof is clear.

$(\Leftarrow)$ Let $x, y \in E$ and $(x \to y) \to x \in F$. By Proposition 1.18 (v), $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$. Then, by Lemma 2.3, F is a filter,$(x \to y) \to ((x \to y) \to y) \in F$. Since F is a positive implicative filter, by Proposition 2.6 (ii), $(x \to y) \to y \in F$.Moreover, by Propositions 1.18 (iii) and 1.19 (i), $(x \to y) \to x \leq y \to x$. Thus, $y \to x \in F$. Since F is a fantastic filter,$((x \to y) \to y) \to x \in F$. Since$(x \to y) \to y \in F$ and F is a filter, $x \in F$. Hence, by Proposition 2.17 (ii), F is an implicative filter of E. ■

3.2 Boolean and prime filters in equality algebras

In this section, we introduce the notions of Boolean and prime filters in equality algebras and investigate some of their properties.

Definition 3.16 *[1]Let E be a bounded lattice equality algebra. A filter F of E is called a Boolean filter if, for all $x \in E, x \vee x\prime \in F$.*

Example 3.17 *Let E be an equality algebra as in Example 3.2. By routine calculations, we can see that $F = \{1, a, b, c\}$ is a Boolean filter of E.*

Theorem 3.18 *[1]Suppose E is a bounded lattice equality algebra and F is a filter of E. Then F is a Boolean filter if and only if F is an implicative filter.*

Proof. ($\Rightarrow$) Let F be a Boolean filter. Then for any $x \in E$, $x \vee x' \in F$. If $x' \to x \in F$, then, by Proposition 1.20 (ii), $(x \vee x') \to x = (x \to x) \wedge (x' \to x) = x' \to x \in F$. Since F is a filter, $x \in F$.

($\Leftarrow$) Suppose F is an implicative filter. Since$x' \wedge x'' \leq x' \leq x' \vee x$, by Proposition 1.21 (ii), we have $(x \vee x')' \to (x \vee x') = (x' \wedge x'') \to (x \vee x') = 1 \in F$. Hence, by Proposition 2.17 (iii), $x \vee x' \in F$. ■

Corollary 3.19 *[1]Suppose E is a bounded equality algebra. Then E is a Boolean algebra if and only if $\{1\}$ is a Boolean filter.*

Proof. From Proposition 2.27 and Theorem 3.18, the proof is clear. ■

Corollary 3.20 *[1]Each Boolean filter of a bounded lattice equality algebra is a positive implicative and a fantastic filter.*

Proof. By Theorems 2.19, 3.12, and 3.18, the proof is clear.The following example shows that the converse of Corollary 3.20 may not be true, in general. ■

Example 3.21 (i) *In Example 2.5, $F = \{1, c\}$ is a fantastic filter, but it is not a Boolean filter. Because, $a \vee a' = a \notin F$.*

(ii) *In Example 2.2, $G = \{1, b\}$ is a positive implicative filter, but it is not a Boolean filter. Because, $a \vee a' = a \notin G$.*

Corollary 3.22 *[1]In any bounded commutative equality algebra, implicative, positive implicative, and Boolean filters coincide.*

Proof. By Corollary 3.6, Theorems 3.15, 1.22, and 3.18, the proof is clear. ■

Definition 3.23 *[1]A proper filter F of E is called a prime filter if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.*

Example 3.24 *In Example 3.2, $\{1, a\}, \{1, b\}$, and $\{1, a, b, c\}$ are prime filters.*

Theorem 3.25 *[1]Let F be a filter of lattice equality algebra E. Then the following statements are equivalent:*

(i) *F is a maximal and Boolean filter,*

(ii) *F is a maximal and positive implicative filter,*

(ii) *$x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for all $x, y \in E$,*

(iv) *F is a prime and Boolean filter,*

(v) *F is a proper filter such that $x \in F$ or $x' \in F$, for every $x \in E$.*

Proof. By Theorems 3.18 and 2.28, the proof of $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$ are clear.

$(i) \Rightarrow (iv)$ Suppose F is not a prime filter. Then there exist$x, y \in E$ such that $x \to y \notin F$ and $y \to x \notin F$. Since $x \leq y \to x$ and F is a filter, $x \notin F$. By Proposition 2.12, F_x is the least filter containing F and x. Also, by assumption, F is a maximal filter, and so $F_x = E$.Thus, $y \in F_x$, and so $x \to y \in F$, which is a contradiction.

$(iv) \Rightarrow (v)$ Let F be a prime and Boolean filter. Then, for any $x \in E$, $x \to x' \in F$ or $x' \to x \in F$. If $x \to x' \in F$, then, by Proposition 1.20 (ii), $(x \vee x') \to x' = (x \to x') \wedge (x' \to x') = x \to x' \in F$. Since F is a Boolean filter, $x' \in F$. By a similar way, if $x' \to x \in F$, then $x \in F$.

$(v) \Rightarrow (i)$ Let F be a proper filter such that satisfies (v). If $x \in F$, then $x \vee x' \in F$. If $x \notin F$, then by (v), $x' \in F$, and so $x \vee x' \in F$. Hence, F is a Boolean filter. Now, we prove that F is a maximal filter. Let G be a proper filter of E such that $F \subseteq G \subsetneqq E$. If $x \in G \setminus F$, then $x' \in F$, and so $x' \in G$. Hence, $0 \in G$, which is a contradiction. ■

Theorem 3.26 *[1]Let F be a proper filter of prelinear equality algebra E. Then the following statements are equivalent:*

(i) *F is a prime filter,*

(ii) *for each $x, y \in E$, if $x \vee y \in F$, then $x \in F$ or $y \in F$,*

(iii) *E/F is a chain or equivalently $\leq_F$ is totally ordered.*

Proof. $(i) \Rightarrow (ii)$ Suppose F is a prime filter and $x \vee y \in F$. Since E is prelinearly, $(x \to y) \vee (y \to x) = 1 \in F$. Since F is prime, $x \to y \in F$, by Proposition 1.20 (i), $(x \vee y) \to y \in F$. Thus, $y \in F$.

$(ii) \Rightarrow (iii)$ Since E is prelinear, by (ii), $x \to y \in F$ or $y \to x \in F$. Thus, F is a prime, and so $[x] \leq_F [y]$ or $[y] \leq_F [x]$. Hence, E/F is a chain.

$(iii) \Rightarrow (ii)$ The proof is clear. ■

Corollary 3.27 *[1]Let F be a proper filter of prelinear equality algebra E. Then F is a prime filter if and only if E/F is a chain.*

Conclusion

In this study, the notions of equality algebra and there proprieties are given. Therefore, we proposed the concepts of (positive) implicative, fantastic, and Boolean filters in equality algebras and presented a several properties of them. The relations between these filters and quotient structures which are constructed via them are given.

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